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Scalar Field Equation in Robertson–Walker Space-Time

Antonio Zecca[1]

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The quantization of the scalar field is reconsidered in some of its basic elements in the context of the Robertson–Walker space-time. The integration of the generalized Klein–Gordon equation is performed by preliminary separation of the equation with the usual separation method. The orthonormal mode solutions are determined by the explicit integration of the resulting angular and radial equations and by standard properties of the time equation. The time evolution given by the standard cosmological model is briefly discussed.

1. INTRODUCTION

The scalar field equation, which is the simplest relativistic field equation associated to spin-0 particles, has been widely studied in connection with the problem of the field quantization in curved space-time (Birrell and Davies, 1982, and references therein; Fulling, 1989). The field equation can be derived by applying to the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\sqrt{-g}\{\nabla_\mu\phi(x)\nabla^\mu\phi(x) - [m^2 + \xi\overline{R}(x)]\phi(x)^2\} \tag{1}$$

[$\phi(x)$ is the scalar field, m is the mass of the field quanta, ∇_μ is the covariant derivative, $\overline{R}(x) = g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar, and ξ is a real numerical factor] the Euler–Lagrange equation

$$\frac{\partial\mathscr{L}}{\partial\phi} - \nabla_\alpha \frac{\partial\mathscr{L}}{\partial(\nabla_\alpha\phi)} = 0 \tag{2}$$

[1] Dipartimento di Fisica dell' Universita' and INFN, Sezione di Milano, I-20133 Milan, Italy.

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which follows from the Hamilton principle relative to the scalar action constructed from $\mathscr{L}$ (e.g., Illge, 1993). One gets the generalized Klein–Gordon equation

$$\nabla_\alpha \nabla^\alpha \phi(x) + [m^2 + \xi \bar{R}(x)]\phi(x) = 0 \tag{3}$$

Corresponding to the solutions ϕ_1, ϕ_2 of equation (3) one also considers the four-vector (the four-current if $\phi_1 = \phi_2$)

$$J_\mu(\phi_1, \phi_2) = -i(\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1) \tag{4}$$

which is divergence-free as a direct consequence of equation (3) and of the reality of ξ. Accordingly, the expression

$$(\phi_1, \phi_2) \equiv \int_\Sigma J_\mu(\phi_1, \phi_2)[-g_\Sigma(x)]^{1/2} n^\mu \, d\Sigma \tag{5}$$

$$\equiv \int_{t=t_0} J_t(\phi_1, \phi_2)(-g_{t_0})^{1/2} \, d_3x \tag{6}$$

(Σ is a spacelike hypersurface of volume element $d\Sigma$ and n^μ is a future-directed unit vector orthogonal to Σ) is independent of the Cauchy surface Σ, as follows from (4) and (5) by Gauss' theorem (Birrell and Davies, 1982) and defines an inner product whose explicit value can be more easily calculated by expression (6). One can then select a complete set of solutions ϕ_α of equation (3) which are orthonormal in the product (5),

$$(\phi_\alpha, \phi_\beta) = \delta_{\alpha\beta}, \qquad (\phi_\alpha^*, \phi_\beta^*) = -\delta_{\alpha\beta}, \qquad (\phi_\alpha, \phi_\beta^*) = 0 \tag{7}$$

and proceed with the covariant quantization of the theory in analogy with the Minkowski-space case. [As is well known, there remains, however, an intrinsic ambiguity in the quantization procedure due to the lack of privileged coordinates (Fulling, 1973; Birrell and Davies, 1982).]

From the above general picture it follows that, in the case of concrete examples of space-time, a central role for covariant quantization is played by the knowledge of the explicit solutions of equation (3). The mentioned procedure has been completely performed in the case of the Robertson–Walker space-time (Bander and Itzykson, 1966; Parker and Fulling, 1974; Ford, 1976); the results are summarized in Birrell and Davies (1982).

It is of some interest to have a unified and simple derivation of the result relative to the case of the Robertson–Walker space-time. This is what will be done in the following sections, where the standard separation method is applied directly to equation (3) without introducing any conformal time parameter. The separated angular and radial equations are integrated explicitly. The general normalization conditions (7) are imposed on the solutions of equation (3) by exploiting a formal property of the separated time equation. The time equation can be given in a closed form by using a conformal time parameter, but its analytical solution is in general difficult.

2. SEPARATION OF THE EQUATION

In the Robertson–Walker space-time whose metric is given by

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \qquad (a = 0, \pm 1) \tag{8}$$

the field equation (3) becomes, in terms of the coordinate derivatives and by using the relation $\Gamma^{\mu}_{\mu\alpha} = (-g)^{-1/2}\partial_\alpha(-g)^{1/2}$,

$$\phi_{tt} - \frac{1 - ar^2}{R^2(t)}\phi_{rr} - \frac{1}{r^2R^2(t)}\left(\phi_{\theta\theta} + \frac{1}{\sin^2\theta}\phi_{\varphi\varphi}\right) + 3\frac{\dot{R}(t)}{R(t)}\phi_t - \frac{\cot\theta}{r^2R^2(t)}\phi_\theta$$
$$+ \frac{3ar^2 - 2}{rR^2(t)}\phi_r + (m^2 + \xi\overline{R}(t))\phi = 0 \qquad (a = 0, \pm 1) \tag{9}$$

where now $\overline{R}(t) = 6[\ddot{R}(t)R(t) + \dot{R}^2(t) + a]/R^2(t)$.

[It must be remarked that equation (3) could be made explicit also by the Newman–Penrose formalism, Newman and Penrose (1962). Even if this method is in general convenient for higher spin values, in the present case it is quite cumbersome. In any event, when applied to equation (3), by the choice of the null tetrad frame already used in other situations (Zecca, 1996), it leads exactly to equation (9).] By setting now

$$\phi(r, \theta, \varphi, t) = T(t)\chi(\theta, \varphi)S(r) \tag{10}$$

we can separate equation (9) to get

$$\ddot{T}R^2 + 3\dot{R}R\dot{T} + [k^2 + m^2R^2 + 6\xi(\ddot{R}R + \dot{R}^2 + a)]T = 0 \tag{11}$$

$$\chi_{\theta\theta} + \frac{1}{\sin^2\theta}\chi_{\varphi\varphi} + \cot\theta\chi_\theta = -\lambda\chi \tag{12}$$

$$(1 - ar^2)S'' + \left(\frac{2}{r} - 3ar\right)S' + \left(k^2 - \frac{\lambda}{r^2}\right)S = 0 \tag{13}$$

k^2, λ are the separation constants relative to the time and angular equations, respectively ($\dot{T} = dT/dt$, $S' = dS/dr$). Equation (12) is the angular part of the

Schrödinger equation in polar coordinates. By assuming the usual regularity conditions on χ for $\theta = 0, \pi$ and $\varphi = 0, 2\pi$, one has therefore

$$\chi(\theta, \varphi) = Y_{lm}(\theta, \varphi), \qquad \lambda = l(l+1), \qquad m = -l, -l+1, \ldots, l$$
$$(l = 0, 1, 2, 3, \ldots) \tag{14}$$

As to the time equation (11), it is independent of a only for $\xi = 0$. In view of making explicit the condition (7), we remark that the Wronskian of the solution $T_k(t)$ satisfies

$$\frac{d}{dt}(T_k\dot{T}_k^* - \dot{T}_kT_k^*) = 3\frac{\dot{R}}{R}(\dot{T}_kT_k^* - T_k\dot{T}_k^*) \tag{15}$$

as a consequence of equation (11), and it can therefore be normalized to

$$T_k\dot{T}_k^* - \dot{T}_kT_k^* = \frac{i}{R^3} \tag{16}$$

With regard to equation (13), it strictly depends on the different values of a, which will be considered separately. Its solutions must also satisfy the condition $S(0) = 0$ for $\lambda \neq 0$, as follows from the equation itself.

3. THE RADIAL EQUATIONS

Case $a = 0$. Equation (13) becomes

$$S'' + \frac{2}{r}S' + \left[k^2 - \frac{l(l+1)}{r}\right]S = 0 \tag{17}$$

whose acceptable solutions are the spherical Bessel functions of the first kind $j_l(kr)$. Then the mode solutions

$$\phi_{klm}(r, \theta, \varphi, t) = T_k(t)Y_{lm}(\theta, \varphi)(2/\pi)^{1/2}kj_l(kr) \tag{18}$$

satisfy the proper orthonormalization conditions

$$\begin{aligned}(\phi_{klm}, \phi_{k'l'm'}) &= \delta_{mm'}\delta_{ll'}\delta(k - k')\\ (\phi_{klm}^*, \phi_{k'l'm'}^*) &= -\delta_{mm'}\delta_{ll'}\delta(k - k')\\ (\phi_{klm}, \phi_{k'l'm'}^*) &= 0\end{aligned} \tag{19}$$

as follows from equations (6), (16), and (18).

The cases $a = \pm 1$ can be reduced by first setting $S = r^lZ(r)$ in Eq. (13), thus obtaining for Z the equation

$$(1 - ar^2)Z'' + \left[\frac{2l+2}{r} - ar(2l+3)\right]Z' - [a(l^2 + 2l) - k^2]Z = 0 \tag{20}$$

In the case $a = 1$, by setting $r = \sin\chi$ $(0 \leq \chi \leq \pi)$ and then $x = (\cos\chi + 1)/2$, equation (20) becomes the hypergeometric equation

$$x(1 - x)Z'' + \left[l + \frac{3}{2} - x(3 + 2l)\right]Z' - (l^2 + 2l - k^2)Z = 0 \quad (21)$$

Acceptable solutions are given by $F(l + 1 + H, l + 1 - H; l + 3/2; x)$, where $H^2 = k^2 + 1$, and one could proceed as in the following case $a = -1$. However, in the present case, it is also possible to select a (complete) countable set of solutions by choosing $l + 1 - H = d$ $(d = 0, 1, 2, \ldots)$ or $H = l + 1 + d = n$ with $n = 1, 2, 3, \ldots$ $(k^2 = n^2 - 1)$. Hence $Z = F(l + 1 - n, l + 1 + n; l + 3/2; x) \cong C^{l+1}_{n-l-1}(\cos\chi)$, C^α_n being the ultraspherical (Gegenbauer) polynomials (Abramovitz and Stegun, 1972). Therefore a complete set of radial modes is given by (Ford, 1976)

$$S_{nl} = 2^l l! \left[\frac{2}{\pi}\frac{(n - l - 1)!\, n!}{(n + l)!}\right]^{1/2} r^l C^{l+1}_{n-l-1}(\cos\chi) \qquad (r = \sin\chi) \quad (22)$$

whose corresponding $\phi_{nlm} = T_n(t)Y_{lm}(\theta, \varphi)S_{nl}(r)$ $(k_n^2 = n^2 - 1)$ satisfy

$$\begin{aligned} (\phi_{nlm}, \phi_{n'l'm'}) &= \delta_{mm'}\delta_{ll'}\delta_{nn'} \\ (\phi^*_{nlm}, \phi^*_{n'l'm'}) &= -\delta_{mm'}\delta_{ll'}\delta_{nn'} \qquad (23) \\ (\phi_{nlm}, \phi^*_{n'l'm'}) &= 0 \end{aligned}$$

In the case $a = -1$, by setting $r = \sinh\chi$ $(0 \leq \chi \leq \infty)$ in equation (20) and then $x = (\cosh\chi + 1)/2$ in the resulting equation, one gets exactly equation (21) with the substitution $k \to ik$. If now $-L^2 = 1 - k^2$, by using the iterated differential formula and a special elementary case of the hypergeometric function (Abramovitz and Stegun, 1970), one gets

$$\begin{aligned} Z &\cong F(l + 1 - il, l + 1 + il; l + 3/2; x) \\ &\cong \left(\frac{d}{d\cosh\chi}\right)^{l+1} F(-iL, iL; 1/2; x) \\ &\cong \left(\frac{d}{d\cosh\chi}\right)^{l+1} \cos L\chi \qquad (24) \end{aligned}$$

By repeated integration by parts on the r variable, one can easily show (Bander and Itzykson, 1966; Dolginov and Toptygin, 1960) that the radial functions

$$S_{Ll}(r) = \left[\frac{\pi}{2} L^2(L^2 + 1) \cdots (L^2 + l^2)\right]^{1/2} r^l \left(\frac{d}{d\cosh\chi}\right)^{l+1} \cos L\chi \qquad (r = \sinh\chi) \quad (25)$$

give the proper orthonormal property of the solutions $\phi_{Llm} = T_L(t)Y_{lm}(\theta, \varphi)S_{Ll}(r)$:

$$(\phi_{Llm}, \phi_{L'l'm'}) = \delta_{mm'}\delta_{ll'}\delta(L - L')$$
$$(\phi^*_{Llm}, \phi^*_{L'l'm'}) = -\delta_{mm'}\delta_{ll'}\delta(L - L') \quad (26)$$
$$(\phi_{Llm}, \phi^*_{L'l'm'}) = 0$$

4. THE TIME EVOLUTION

The time equation can be solved directly in the case of static Robertson–Walker space-time ($R = R_0$). The normalized solutions of equation (11) are in this case

$$T_k(t) = (2\omega_k R_0^3)^{-1/2} e^{-i\omega_k t} \quad (27)$$

where

$$\omega_k = \frac{1}{R_0}\sqrt{m^2 R_0^2 + k^2 + 6a\xi} \quad (28)$$

In the Einstein universe ($a = 1$), by choosing also $\xi = 1/6$, the discrete modes of the previous section have frequencies

$$\omega = \omega_n = \frac{1}{R_0}\sqrt{m^2 R_0^2 + n^2}, \qquad n = 1, 2, 3, \ldots \quad (29)$$

with a degeneracy of n^2 as pointed out by Ford (1976).

Equation (11) can be solved in principle by giving the explicit form of $R(t)$. An analytical solution of equation (11) is not easy in general nor in the standard cosmology, where $R(t)$ is given in a parametric form. Indeed, it is well known that (e.g., Kolb and Turner, 1990) the motion of the open and closed standard cosmological models is respectively of the form

$$R(t) = A(\cosh\psi - 1), \qquad t = B(\sinh\psi - \psi), \qquad \psi \geq 0 \qquad (a = -1) \quad (30)$$

$$R(t) = C(1 - \cos\theta), \qquad t = D(\theta - \sin\theta), \qquad 0 \leq \theta \leq 2\pi \qquad (a = 1) \quad (31)$$

A, B, C, D are positive constants depending on the physical content of the theory. Some simplification can be obtained by using the conformal time parameter

$$\tau = \int_0^t \frac{dt}{R(t)} \quad (32)$$

in terms of which equation (11) becomes

$$T'' + 2\frac{R'}{R}T' + \left[k^2 + m^2R^2 + 6\xi\left(\frac{R''}{R} + a\right)\right]T = 0 \tag{33}$$

where now $R = R(\tau)$ and $' = d/d\tau$. If the integral (32) is performed by using the expressions (30) and (31), one gets $\tau = (B/A)\psi$ and $\tau = (D/C)\theta$, respectively. Therefore, in the standard model

$$R(\tau) = A\left(\cosh\frac{A}{B}\tau - 1\right) \qquad (a = -1) \tag{34}$$

$$R(\tau) = C\left(1 - \cos\frac{C}{D}\tau\right) \qquad (a = 1) \tag{35}$$

When inserted in equation (33) these expressions give a closed form of the time equation that not only could be solved by a numerical integration, but that also gives information on some limiting physical situations. As an example, the open, massless case ($m = 0$, $a = -1$, $\xi = 1/6$) gives, for large t ($\tau >> 1$), the equation

$$T'' + 2\frac{A}{B}T' + \left(k^2 + \frac{A^2}{B^2} - 1\right)T = 0 \tag{36}$$

leading to the solution

$$T \cong T_L(t) = \frac{1}{A}(2L)^{-1/2}\exp\left[-\left(\frac{A}{B} + iL\right)\tau\right] \qquad (\tau >> 1, \quad L^2 = k^2 - 1) \tag{37}$$

which satisfies the normalization condition (16).

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